FIELD THEORETICAL BACKGROUND FOR THERMAL PHYSICS

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ABSTRACT

Techniques of zero-temperature field theory that have found application in the analysis of field theory at finite temperature are revisited. Specifically, several of the results that are discussed are relevant to the study of symmetry-changing phase transitions and high temperature QCD, which today are among the most actively investigated problems in finite temperature field theory.

1. Overview

Most of the techniques used in finite temperature field theory were first developed for analysis of field theory at zero temperature. The purpose of these pedagogical lectures is to revisit (a part of) this field theoretical background.

We start, in Secs. 2 and 3, by discussing techniques for studying the phase structure of a field theory using Green's functions generating functionals. These techniques are receiving renewed interest because of their role in numerous recent studies of temperature-induced symmetry-changing phase transitions, which have followed the suggestion that the observed baryon number asymmetry might have originated at the electroweak phase transition.

Also relevant to the investigation of temperature-induced phase transitions is the formalism of non-equilibrium quantum field theories, which we discuss in Sec.4.

Finally, in Sec. 5 we review Chern-Simons theory, which has recently found application in finite temperature field theory because the Chern-Simons eikonal gives the generating functional of the *hard thermal loops* in QCD.

2. Phase Structure and Generating Functionals

2.1. Effective Action

An important aspect of a physical system is its phase structure (what are the available phases, in what phase the system is at a given time, what is the nature of the phase transitions ....). Let us assume, at least at the beginning, that one of the fields in the theory (Φ) is a scalar and study the phase structure in terms

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of the thermal expectation of that scalar field, defined by (we use units such that $\hbar = c = k_{Boltzman} = 1$)

$$< \Phi >\equiv \frac{tr\ e^{-\beta H}\Phi}{tr\ e^{-\beta H}} \ , \tag{1}$$

where β is the inverse temperature $1/T$. At zero temperature one is interested in the zero-temperature limit of $< \Phi >$, *i.e.* the vacuum expectation value $< 0|\Phi|0 >$. If one could solve the theory completely these expectation values could be evaluated explicitly from their definitions. Unfortunately there are only a few quantum field theories that can be solved completely (and they are not very interesting from the physical point of view), so that the answer to the problem at hand can only be approached somewhat indirectly by setting up additional formalism, with the hope that a resonable approximation scheme emerges.

We use an approach based on generating functionals: the effective action, the effective energy, and the effective potential. This was first proposed by Jona-Lasinio[1], and later developed in Refs.[2 – 5]. (For a review of this subject see Ref.[6].) For definiteness, we work at zero-temperature; however, the finite temperature generalizations can be easily obtained following the discussion in Sec.3.

The first step is the introduction of the "partition function" $Z(J)$

$$Z(J) \equiv< 0|\mathcal{T} \exp i \int_x J(x)\Phi(x)|0 > \ , \tag{2}$$

where $\mathcal{T}$ is the time-ordering operator.

$Z(J)$ is a derived quantity, a quantity external to the theory, but its importance is due to the fact that $Z(J)$ is the generating functional for Green's functions:

$$G_n(x_1, ..., x_n) \equiv< 0|\mathcal{T}\Phi(x_1).....\Phi(x_n)|0 >= \left[\frac{\delta^n}{i^n \delta J(x_1)....J(x_n)} Z(J)\right]_{J=0} \ . \tag{3}$$

In as much as all the physical information on a quantum field theory is encoded in the Green's functions, one can study the theory just using the generating functional $Z(J)$.

The generating functional for connected Green's functions $W(J)$ is given by

$$W(J) \equiv -i \ln Z(J) \ . \tag{4}$$

The same physical information encoded in the Green's functions is organized more economically in the connected Green's functions; in fact, once the connected ones are known the complete Green's functions can be reconstructed.

The effective action $\Gamma(\phi)$ is defined as the functional Legendre transform of $W(J)$:

$$\Gamma(\phi) = W(J) - \int_x \ \phi(x)J(x) \ , \tag{5}$$

where

$$\phi(x) \equiv \frac{\delta W(J)}{\delta J(x)} \ , \tag{6}$$

and in (5) J is expressed in terms of ϕ using (6). [$\Gamma(\phi)$ is sometimes referred to as the "quantum action"; in the limit $\hbar \to 0$ it reproduces the classical action.]

Evidently

$$\frac{\delta\Gamma(\phi)}{\delta\phi(x)} = -J(x) \ , \tag{7}$$

and, as a consequence, physical solutions, which require vanishing J, correspond to the stationary points of Γ.

From the definitions that we have introduced it is clear that

$$< 0|\Phi(x)|0 >= [\phi(x)]_{J=0} \equiv \phi_0(x), \tag{8}$$

and therefore using Eq.(7) one finds that the vacuum expectation value of Φ is given by the value of $\phi(x)$ which stationarizes $\Gamma(\phi)$:

$$\left[\frac{\delta\Gamma(\phi)}{\delta\phi(x)}\right]_{\phi(x)=\phi_0(x)} = 0 \to \phi_0(x) =< 0|\Phi(x)|0 > \ . \tag{9}$$

We have replaced the problem of evaluating the vacuum expectation value of a quantum field by the problem of finding the stationary points of a classical functional. Such a problem has a long tradition in physics and we can use the numerous approximation techniques that have been developed over the years.

Finally we note that $\Gamma(\phi)$ is the generating functional of the one particle irreducible* (1PI) Green's functions:

$$\left[\frac{\delta^n\Gamma(\phi)}{\delta\phi(x_1).....\delta\phi(x_n)}\right]_{\phi(x)=\phi_0(x)} = G_n^{1PI}(x_1, ..., x_n) \ . \tag{10}$$

These contain all the physical information encoded in the Green's functions in an even more economical format than the connected Green's functions.

2.2. Effective Potential and Effective Energy

If one is interested in all the physical information in a quantum field theory, it is necessary to study the complete effective action, as a functional of the arbitrary background field $\phi(x)$. However, when one is only interested in the vacuum expectation value $\phi_0(x) =< 0|\Phi(x)|0 >$, one can use any *a priori* information available on $\phi_0(x)$ to simplify the analysis.

If it is expected that translation invariance is not broken, as is usually the case, one can assume that $\phi_0(x)$ be independent of the space-time point: $\phi_0(x) = constant = \phi_0$. With this hypothesis, in order to determine ϕ_0 it is sufficient to study the effective action as a function of a constant background ϕ. However, when ϕ is constant an infinite volume factor arises from the space-time integrations, and therefore, rather than working with the effective action, we introduce the effective potential $V(\phi)$, defined by

$$V(\phi) \int_x \equiv -[\Gamma(\phi)]_{\phi=constant} \ . \tag{11}$$

* A graph is said to be "one particle irreducible" if it does not become disconnected upon opening any one line; otherwise it is "one particle reducible".

The effective potential is the generating functional of the 1PI Green's functions at zero energy and momentum*, and in the $\hbar \to 0$ limit the effective potential reproduces the classical potential. Clearly from Eqs.(9) and (11) it follows that for translationally invariant theories the vacuum expectation value can be determined using the effective potential:

$$\left[\frac{\delta V(\phi)}{\delta \phi}\right]_{\phi=\phi_0} = 0 \to \phi_0 =< 0|\Phi(x)|0 > \ . \tag{12}$$

An intermediate possibility between evaluating the effective action as a functional of a space-time dependent background $\phi(x)$ and the much simpler case of a constant background, is the evaluation of the effective action for static (time-independent, position-dependent) background $\phi(\mathbf{x})$. In this circumstance, the time integration in $\Gamma(\phi)$ leads to an overall (infinite) time-interval factor; it is therefore convenient to introduce the effective energy $E(\phi)$, defined by

$$E(\phi) \int dx_0 \equiv -[\Gamma(\phi)]_{\phi=static} \ . \tag{13}$$

The effective energy is the generating functional for 1PI Green's functions at zero energy, in the $\hbar \to 0$ limit it reproduces the classical energy, and in time translation-invariant theories it can be used to determine the vacuum expectation value using:

$$\left[\frac{\delta E(\phi)}{\delta \phi(\mathbf{x})}\right]_{\phi(\mathbf{x})=\phi_0(\mathbf{x})} = 0 \to \phi_0(\mathbf{x}) =< 0|\Phi(x)|0 > \ . \tag{14}$$

2.3. Loop Expansion

In the search for the stationary points of $\Gamma(\phi)$, the choice of a strategy for evaluating $\Gamma(\phi)$ is very important. One commonly used technique is the approximation of $\Gamma(\phi)$ based on the following known[3,4] expansion**

$$\Gamma(\phi) = I(\phi) - \frac{i}{2} tr \ln D + \Gamma_2(\phi) \ . \tag{15}$$

$I(\phi)$ is the classical action. The functional operator $D^{-1}(\phi; x, y)$ is defined by:

$$D^{-1}(\phi; x, y) = \frac{\delta^2 I(\phi)}{\delta \phi(x) \delta \phi(y)} \ . \tag{16}$$

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* Note that in general the Green's functions at zero energy and momentum are not physical, though they might have physical relevance in particular contexts. The poles and the residues at the poles of the Green's functions are physical.

** The expansion is formally in powers of $\hbar$ but this doesn't explicitly appear in Eq.(15) because in our units $\hbar = 1$.

The term $\frac{i}{2}[tr \ln D]$, where the trace and the logarithm are taken in the functional sense, is the one loop contribution [it is $O(\hbar)$]. $\Gamma_2(\phi)$ is computed as follows. In the classical action $I(\Phi)$, shift the quantum field $\Phi(x)$ by a c-number field $\phi(x)$. $I(\Phi+\phi)$ contains terms cubic and higher in Φ that define an interaction part $I_{int}(\phi;\Phi)$ whose vertices depend on ϕ. $\Gamma_2(\phi)$ is the sum of all the two-or-more-loop 1PI vacuum graphs in the theory with vertices given by $I_{int}(\phi;\Phi)$ and propagators taken to be $D(\phi;x,y)$.

For example, for the $\lambda\Phi^4$ scalar theory

$$I(\Phi) = \int_x \left[\frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) + \frac{a}{2}\Phi^2 - \frac{\lambda}{4!}\Phi^4\right] , \tag{17}$$

$$D^{-1}(\phi;x,y) = (-\Box + a - \frac{\lambda}{2}\phi^2(x))\delta^4(x-y) , \tag{18}$$

$$I_{int}(\phi;\Phi) = \int_x \left[-\frac{\lambda}{6}\phi\Phi^3 - \frac{\lambda}{4!}\Phi^4\right] . \tag{19}$$

The diagrams which contribute to $\Gamma_2(\phi)$ for the $\lambda\Phi^4$ scalar theory are shown in Fig.1 up to three loops. The lines represent $D(\phi;x,y)$ and, as indicated by Eq.(19), there are three-point and four-point vertices.

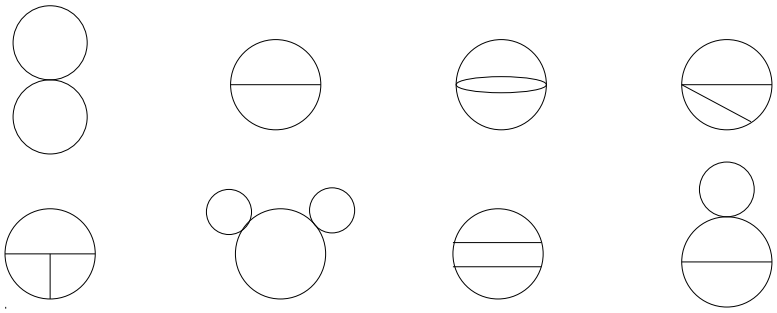

Fig. 1

A loop expansion for the effective potential can obviously be obtained by evaluating Eq.(15) for constant ϕ and extracting a (infinite) volume factor:

$$V(\phi) = V_{cl}(\phi) + \frac{i}{2}\int_k \ln D(\phi;k) + V_2(\phi) , \tag{20}$$

where $V_{cl}(\phi)$ is the classical potential, $D(\phi;k)$ is the Fourier-transform of $D(\phi;x,y)$ at constant ϕ

$$D(\phi;x,y) \equiv \int_k e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} D(\phi;\mathbf{k}) , \tag{21}$$

and $V_2(\phi)$ is given by

$$V_2(\phi)\int_x \equiv -[\Gamma_2(\phi)]_{\phi=constant} . \tag{22}$$

Analogously, the loop expansion for the effective energy is obtained by evaluating Eq.(15) for static ϕ and extracting a (infinite) time-interval factor.

2.4. Selective Summations and Effective Action for Composites

It is very difficult to carry out the loop expansion beyond two loops. It is however possible to perform selective summations of higher loop graphs. One way to systematize such summations is the large N method, in which one considers an N-component field and uses the fact that in the large N limit some multi-loop graphs give greater contributions than others. For example in the $\lambda\Phi^4$ N-component scalar theory the leading multi-loop contributions come from *bubble-graphs* (see Fig.2).

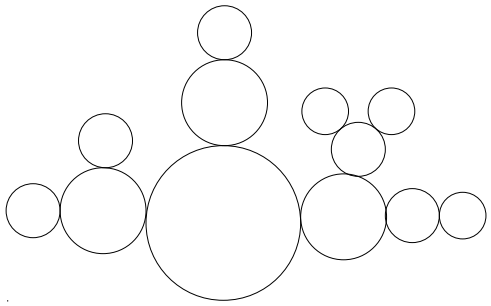

Fig. 2

Another way to perform systematic selective summations makes use of the effective action for composites[7]. This generalization of the effective action was primarily introduced to seek the phase structure at the level of the two-point function*, in addition to, or instead of, the one-point function. Moreover, it can also be used to rexpress the ordinary effective action in a way that is more suitable for selective summations.

One arrives at the definition of the effective action for composites following a path very analogous to the one that leads to the ordinary effective action. The starting point is a new partition function that depends on a bilocal source $K(x,y)$, in addition to the local source $J(x)$:

$$Z(J,K) \equiv e^{iW(J,K)} \equiv < 0|\mathcal{T} \exp i\left[\int_x J(x)\Phi(x) + \frac{1}{2}\int_{x,y} \Phi(x)K(x,y)\Phi(y)\right]|0> \ , \tag{23}$$

where we have also introduced $W(J,K)$. A possible expectation value of the quantum field $\Phi(x)$ is $\phi(x)$; this is again defined as the functional derivative of W with respect to J

$$\phi(x) \equiv \frac{\delta W(J,K)}{\delta J(x)} \ , \tag{24}$$

but now one also introduces $G(x,y)$, defined by

$$\frac{\delta W(J,K)}{\delta K(x,y)} \equiv \frac{1}{2}\phi(x)\phi(y) + \frac{1}{2}G(x,y) \ , \tag{25}$$

* For example Fermi fields do not have a vacuum expectation value, but the fermion propagator can show spontaneous chiral symmetry breaking through mass generation.

which is a possible expectation value of the connected part of the two point function.

The effective action for composites $\Gamma(\phi, G)$ is defined as the double Legendre transform of $W(J, K)$

$$\begin{aligned}\Gamma(\phi, G) = W(J, K) - \int_x \phi(x) J(x) \\ - \frac{1}{2}\int_{x,y} \phi(x) K(x, y)\phi(y) - \frac{1}{2}\int_{x,y} G(x, y)K(y, x) \ ,\end{aligned} \tag{26}$$

where J and K are to be expressed in terms of ϕ and G using (24) and (25).

One can show[7] that $\Gamma(\phi, G)$ is the generating functional in ϕ for two-particle irreducible* (2PI) Green's functions, expressed in terms of the full propagator G.

From Eq.(26) it is evident that

$$\frac{\delta\Gamma(\phi, G)}{\delta\phi(x)} = -J(x) - \int_y K(x, y)\phi(y) \ , \tag{27}$$

$$\frac{\delta\Gamma(\phi, G)}{\delta G(x, y)} = -\frac{1}{2}K(x, y) \ . \tag{28}$$

Since physical processes correspond to vanishing sources J and K, Eqns.(27) and (28) imply that $\Gamma(\phi, G)$ is physical at its stationary points.

Eq.(28) also indicates the relation between $\Gamma(\phi, G)$ and the ordinary effective action; in fact, as it is clear from its definition, $\Gamma(\phi, G)$ must reproduce $\Gamma(\phi)$ when $K = 0$, and using Eq.(28) one finds that

$$\Gamma(\phi) = \Gamma(\phi, G_0) \ , \tag{29}$$

when G_0 satisfies the stationary requirement

$$\left[\frac{\delta\Gamma(\phi, G)}{\delta G(x, y)}\right]_{G=G_0} = 0 \ . \tag{30}$$

A loop expansion is also available[7] for the effective action for composites

$$\Gamma(\phi, G) = I_{cl}(\phi) - \frac{i}{2}tr[\ln G - D^{-1}G] + \Gamma_2(\phi, G) \ , \tag{31}$$

where $\Gamma_2(\phi, G)$ is given by the sum of the two-or-more-loop 2PI vacuum graphs in the theory with vertices given by $I_{int}(\phi; \Phi)$ and propagators set equal to $G(x, y)$.

For $G = G_0$ Eq.(31) gives a new expansion for the ordinary effective action. Moreover, using Eq.(31) the gap equation (30) which determines G_0 can also be expressed in terms of a loop expansion as follows

$$G_0^{-1}(x, y) = D^{-1}(x, y) - 2i\left[\frac{\delta\Gamma_2(\phi, G)}{\delta G(x, y)}\right]_{G=G_0} \ . \tag{32}$$

* A graph is said to be "two particle irreducible" if it does not become disconnected upon opening any two lines; otherwise it is "two particle reducible".

Eq.(32) gives the relation between G_0 and the *tree-level propagator* D, and can therefore be used to express the diagrams of the expansion (31) in terms of diagrams of the ordinary expansion (15). For example, for the $\lambda\Phi^4$ scalar theory it is easy to show that the summation of all the contributions of the bubble graphs to the ordinary effective action $\Gamma(\phi)$ corresponds to the solution of the Eqs.(29)-(30) when $\Gamma_2(\phi, G)$ is approximated by its only graph of $O(\lambda)$, *i.e.* the double-bubble graph in Fig.1. This bubble-resummation has been used in a recent study of the temperature-induced phase transition of the $\lambda\Phi^4$ scalar theory[8]. In general the solution of Eqs.(29)-(30) with the approximation of $\Gamma_2(\phi, G)$ by a finite sum of its graphs corresponds to the selective summation of an infinite class of "ordinary graphs" [graphs of the ordinary expansion (15)].

When interested in translation-invariant (time translation-invariant) solutions one can perform selective summations of multi-loop graphs which contribute to the effective potential (energy). These summations may be performed systematically using the effective potential for composites $V(\phi, G)$ and the effective energy for composites $E(\phi, G)$, which, in analogy with Eqs.(11)-(13), are defined in terms of the effective action for composites by

$$V(\phi, G) \int_x \equiv -[\Gamma(\phi, G)]_{translation-invariant} \ , \tag{33}$$

$$E(\phi, G) \int dx_0 \equiv -[\Gamma(\phi, G)]_{time\, translation-invariant} \ . \tag{34}$$

2.5. Variational Interpretation and Variational Approximations

We have seen that, in the formalism of these generating functionals, physical solutions satisfy variational equations which stationarize the functionals. In this subsection we shall discuss the fact that the functionals themselves have a variational interpretation. The knowledge of these variational interpretations is important, especially because they can suggest approximations.

Let us start with the effective energy $E(\phi)$. It can be shown that $E(\phi)$ is the minimal expectation value of the Hamiltonian in normalized states which are constrained so that the expectation value of the quantum field $\Phi(x)$ in those states is equal to $\phi(\mathbf{x})$, *i.e.*

$$\begin{aligned} E(\phi) = & min < \Psi|H|\Psi > \ , \\ < \Psi|\Psi > = 1 \ , \quad & < \Psi|\Phi(x)|\Psi > = \phi(\mathbf{x}) \ . \end{aligned} \tag{35}$$

This result can be immediately related to the general quantum mechanical principle that in searching for the solutions of a static problem one should minimize the expectation value of the Hamiltonian. Using the effective energy this principle is implemented in two steps: one first minimizes the expectation value of the Hamiltonian with the constraint $< \Psi|\Phi|\Psi > = \phi$, and then the resulting ϕ-dependent functional (the effective energy) is minimized with respect to ϕ.

For the effective potential $V(\phi)$ a similar variational interpretation and similar

considerations apply; one finds that

$$\begin{aligned} V(\phi) =& min\left[\frac{<\Psi|H|\Psi>}{\int d\mathbf{x}}\right] , \\ <\Psi|\Psi>=1 \ , \quad & <\Psi|\Phi(x)|\Psi>=\phi \ . \end{aligned} \tag{36}$$

Also the effective potential and the effective energy for composites have a variational interpretation. For example to obtain $E(\phi, G)$ one minimizes the expectation value of the Hamiltonian on normalized states which satisfy two constraints:

$$\begin{aligned} E(\phi, G) =& min <\Psi|H|\Psi> , \\ <\Psi|\Psi>=1 \ , \quad & <\Psi|\Phi(x)|\Psi>=\phi(\mathbf{x}) \ , \\ <\Psi|\Phi(x)\Phi(y)|\Psi>|_{x_0=y_0} &= \phi(\mathbf{x})\phi(\mathbf{y}) + G(\mathbf{x},\mathbf{y}) \ . \end{aligned} \tag{37}$$

The variational definitions for the effective potential and the effective energy are related to the general quantum mechanical variational principle for static problems. Correspondingly, the variational definition for the effective action $\Gamma(\phi)$ can be derived from the Dirac variational principle[9] for time-dependent problems, which instructs minimizing the following quantity

$$\int dt <\Psi, t|i\partial_t - H|\Psi, t> \equiv \int dt d\mathbf{x}\left[\Psi^*(t,\mathbf{x})(i\partial_t - H)\Psi(t,\mathbf{x})\right] \tag{38}$$

with the normalization constraint

$$<\Psi, t|\Psi, t> \equiv \int d\mathbf{x}\Psi^*(t,\mathbf{x})\Psi(t,\mathbf{x}) = 1 \ . \tag{39}$$

$\Gamma(\phi)$ can be variationally defined as follows

$$\Gamma(\phi) = \ min\left[\int dt <\Psi_-, t|i\partial_t - H|\Psi_+, t>\right] \tag{40}$$

where the states $|\Psi_+, t>$ and $|\Psi_-, t>$ are constrained so that

$$\begin{aligned} <\Psi_-, t|\Phi(\mathbf{x})|\Psi_+, t> &= \phi(t,\mathbf{x}) \\ <\Psi_-, t|\Psi_+, t> &= 1 \\ \lim_{t\to\pm\infty} |\Psi_\pm, t> &= |0> \ . \end{aligned} \tag{41}$$

The variational definition of the effective action for composites $\Gamma(\phi, G)$ is analogous, but with the addition of a constraint for $<\Psi_-, t|\Phi(\mathbf{x})\Phi(\mathbf{y})|\Psi_+, t>$.

The variational interpretation of our generating functionals naturally leads to some variational approximations. These approximations are best formulated in the Schrödinger picture for quantum field theory[10,11], in which states are described by wave functionals $\Psi(\phi)$ of a c-number field $\phi(\mathbf{r})$, the inner product is defined by functional integrals

$$<\Psi_1|\Psi_2> \sim \int D\phi\Psi_1^*(\phi)\Psi_2(\phi) \ , \tag{42}$$

the action of the field on the state is by multiplication

$$\Phi|\Psi>\sim \phi\Psi(\phi)\ , \tag{43}$$

and the action of the canonical momentum on the state is by functional differentiation

$$\Pi|\Psi>\sim \frac{1}{i}\frac{\delta}{\delta\phi}\Psi(\phi)\ . \tag{44}$$

In this formalism one finds the nice result that if $\Psi_G(\phi)$ is a Gaussian functional of the form

$$\Psi_G(\phi)\equiv \exp\left[-\frac{1}{4}\int d\mathbf{x}\ d\mathbf{y}\ (\phi(\mathbf{x})-\phi_0(\mathbf{x}))\ G^{-1}(\mathbf{x},\mathbf{y})\ (\phi(\mathbf{y})-\phi_0(\mathbf{y}))\right]\ , \tag{45}$$

then the expectation of the Hamiltonian in $\Psi_G(\phi)$ automatically gives

$$<H>_G=\int D\phi\Psi_G^*(\phi)H\Psi_G(\phi)=E^{bubble}(\phi_0,G)\ , \tag{46}$$

where $E^{bubble}(\phi_0,G)$ is the approximation of the effective energy for composites which includes only the $O(\lambda)$ contribution to $E_2(\phi_0,G)$. Therefore by stationarizing $<H>_G$ with respect to variations of G one obtains the "bubble-resummed" effective energy.

2.6. Comments and Cautions

Some caution is necessary in using the generating functionals that we have discussed. Let us start by analyzing some features of the effective potential* in the simple $\lambda\Phi^4$ scalar theory. Including only the classical and the one loop term one has (see Sec.2.3)

$$V(\phi)\simeq V_{cl}(\phi)+\frac{1}{2}\int_k \ln[k^2-a+\frac{\lambda}{2}\phi^2]\ , \tag{47}$$

where we have set up a Euclidean evaluation of the one loop contribution. (N.B. We are not doing Euclidean field theory. We simply evaluate a Minkowski space-time integral using a continuation to Euclidean space.)

The first important observation is that the integral in Eq.(47) is clearly divergent, and therefore a regularization/renormalization procedure is needed. It turns out that this renormalization can be performed in a rather standard fashion[2-5].

Eq.(47) also indicates that the effective potential is not necessarily real and convex (if $a>0$ the effective potential in Eq.(47) is not convex and has an imaginary part for small ϕ), and therefore it cannot be interpreted as the free energy of the system. It has been suggested[12] that the imaginary part of the effective potential might carry information on the decay properties of the system. The possible non-convexity of the effective potential is important in the Legendre transform procedure[13]; in fact, it may not be possible to find a unique solution for $J(\phi)$ in Eq.(6), because, *e.g.*, there are multiple roots. In these cases, if one is interested in the free energy, a

* Clearly the issues discussed in this section also concern the effective action and the effective energy, but for definiteness we base the presentation on the case of the effective potential.

Maxwell construction can be used, but then the resulting effective potential will not be the generating functional for the Green's functions.

Concerning the effective potentials of gauge theories there is obviously an additional issue to be addressed: gauge invariance. In a gauge theory the effective potential, besides depending on a background scalar field (and its complex conjugate), can also depend on a background gauge field: $V(A_\mu, \phi, \phi^*)$. There are two ways in which the effective potential may fail to be gauge invariant. First it may not be invariant against gauge transformations of A_μ, ϕ, ϕ^*. Second, even if the effective potential is invariant against gauge transformations of A_μ, ϕ, ϕ^*, it may still depend on the gauge choice made in quantization; for example in covariant gauges it may depend on the coefficient ξ of the gauge fixing term $\xi(\partial_\mu A^\mu)^2$. In general the effective potential does depend on ξ.

The gauge dependence of the effective potential can be understood in terms of the fact that off-shell Green's functions are not physical and are not gauge invariant. Although the full effective potential need not be a physical quantity in a gauge theory, gauge invariant physical information can be obtained from it, for example at the stationary points. However, in many cosmological models the detailed profile of the effective potential, even away from the stationary points, plays an important role (slow rolling down out of equilibrium). Therefore for these models the physical interpretation of the full effective potential is an important issue.

3. Finite Temperature Relativistic Quantum Field Theory

The observable properties of a relativistic quantum field theory at nonzero temperature* can be extracted from the thermal Green's functions, defined by

$$G_\beta(x_1, ..., x_n) \equiv < \mathcal{T}\Phi(x_1).....\Phi(x_n) > = \frac{tr\ e^{-\beta H}\mathcal{T}\Phi(x_1).....\Phi(x_n)}{tr\ e^{-\beta H}} \ . \tag{48}$$

Let us consider the two-point thermal Green's function $G_\beta(x_1, x_2)$. The Dyson-Schwinger equations of motion for $G_\beta(x_1, x_2)$ follow from the Heisenberg equations of motion like in the ordinary zero temperature case

$$\Box_1 G_\beta(x_1, x_2) = -i\delta(x_1 - x_2) + < \mathcal{T}\Box\Phi(x_1)\Phi(x_2) > \ . \tag{49}$$

The structure of Eq.(49) is exactly the same as in the zero temperature case. The difference between the thermal Green's function and the zero temperature Green's function is merely in the boundary conditions, which determine the unique, physically appropriate solution to the differential equation. One can therefore follow the same method of solution to Eq.(49) used at zero temperature, except for what concerns the boundary conditions.

For definiteness and simplicity, we illustrate the methods of solution of Eq.(49) in the case of bosonic fields, for which the boundary conditions are

$$G_\beta(x_1, x_2)|_{x_1^0=0} = G_\beta(x_1, x_2)|_{x_1^0=-i\beta} \ . \tag{50}$$

* For a review, see Refs.[14, 15].

These follow from the cyclicity of trace, which defines G_β, see Eq.(48). [Note that Eq.(49) is in real time, but the solution is required to be periodic in imaginary time.] We also assume that, as it happens in most cases of physical interest, translation invariance is not broken, and therefore

$$G_\beta(x_1, x_2) = G_\beta(x_1 - x_2, 0) \equiv G_\beta(x_1 - x_2) \ . \tag{51}$$

Because of the translation invariance it is convenient to go to conjugate space.

In the "real time formalism" one introduces the Fourier integral transform $G_\beta(p)$ of $G_\beta(x_1 - x_2)$:

$$G_\beta(x_1 - x_2) = \int_p e^{-ip(x_1 - x_2)} G_\beta(p) \ . \tag{52}$$

It then follows simply from the fact that the field $\Phi(x)$ is an operator which verifies causal commutation relations, that $G_\beta(p)$ has the following integral representation

$$\begin{aligned} G_\beta(p) =& i \int_{-\infty}^{\infty} \frac{dp_0'}{(2\pi)} \frac{\rho_\beta(p_0', \mathbf{p})}{p_0 - p_0' + i\epsilon} + f(p_0)\rho_\beta(p_0, \mathbf{p}) \\ =& i \int_{-\infty}^{\infty} \frac{dp_0'}{(2\pi)} \rho_\beta(p_0', \mathbf{p}) \left[\frac{1 + f(p_0')}{p_0 - p_0' + i\epsilon} - \frac{f(p_0')}{p_0 - p_0' - i\epsilon} \right] \ , \end{aligned} \tag{53}$$

where f is the bosonic distribution function

$$f(E) = \frac{1}{e^{\beta E} - 1} \ , \tag{54}$$

and ρ_β is a density function to be determined using Eq.(49).

In the lowest order in perturbation theory the density function is known,

$$\rho_\beta^{free} = 2\pi\epsilon(p_0)\delta(p^2 - m^2) \ , \tag{55}$$

and correspondingly the thermal 2-point Green's function is given by

$$G_\beta^{free}(p) = \frac{i}{p^2 - m^2 + i\epsilon} + \frac{2\pi}{e^{\beta E} - 1} \delta(p^2 - m^2) \ . \tag{56}$$

[Here $\epsilon(p_0) \equiv \theta(p_0) - \theta(-p_0)$, and θ is the unit step function, vanishing for negative argument.] Note that G_β^{free} satisfies the free-field Schwinger-Dyson equation $(p^2 - m^2)G^{free} = i$, and the temperature dependence is confined in the term, proportional to $\delta(p^2 - m^2)$, in which boundary conditions are encoded.

As an alternative to the real time formalism one can work in the "imaginary time formalism", in which the Green's functions are considered in imaginary time. As a consequence of the periodicity in imaginary time described in Eq.(50), in this formalism the momentum space two-point thermal Green's functions are the $G_\beta(\omega_n, \mathbf{p})$ defined by Fourier series and integral transforms

$$[G_\beta(x)]_{x_0 = -ix_4} = \frac{i}{\beta} \sum_n e^{-\omega_n x_4} \int \frac{d\mathbf{p}}{(2\pi)^3} e^{i\mathbf{px}} G_\beta(\omega_n, \mathbf{p}) \ , \tag{57}$$

where

$$\omega_n = i\frac{2\pi n}{\beta} \ . \tag{58}$$

In Eq.(57) there is a Fourier series for the time-like component and an ordinary Fourier integral transform for the space-like components.

The same analysis which leads to the spectral representation of the real time Green's functions also leads to the following spectral representation for the imaginary time Green's functions:

$$G_\beta(\omega_n, \mathbf{p}) = i\int_{-\infty}^{\infty} \frac{dp_0'}{(2\pi)} \frac{\rho_\beta(p_0', \mathbf{p})}{\omega_n - p_0'} \ , \tag{59}$$

which involves the same spectral function ρ_β present in Eq.(53), and the same denominator structure, but now p_0 is substituted by the imaginary discrete quantity ω_n and therefore, because the denominator never vanishes, there is no need for $i\epsilon$-prescriptions.

The imaginary time Green's function which corresponds to the spectral function ρ_β^{free} is

$$G_\beta^{free}(\omega_n, \mathbf{p}) = \frac{i}{\omega_n^2 - \mathbf{p}^2 - m^2} \ , \tag{60}$$

which looks like a free zero temperature Green's function, except that p_0 has been replaced by ω_n.

Perturbation theory has its most physical realization in the real time formalism, however some technical problems are encountered. The structure of perturbation theory is the same as at zero temperature because the structure of the equations is the same, but it is difficult to implement the boundary conditions. Moreover, the Green's functions involve a δ-function and therefore in high order calculations one encounters ambiguous products of δ-functions at the same point. (Only the one-loop results can be obtained rather easily in the real time formalism because no products of δ-functions at the same point can occur.) Obviously these difficulties can be overcome by doing things carefully, but it is very hard to find algorithms for perturbative calculations in the real time formalism. A systematic algorithmic prescription is given by the *time path approach*[14], but involves mathematical elaborations that obscure physical content: a specific closed time contour in the complex plane is chosen, and one uses both time- and anti-time-ordered Green's functions with an effective doubling of the degrees of freedom.

Perturbation theory is not pathologic in the imaginary time formalism. In fact, one can introduce Feynman rules in complete analogy to zero temperature, except for the substitution in the propagators of the continuous variable p_0 by the discrete ω_n, and the substitution of the integration over a loop four-momentum by an integration over the three spatial components of the momentum and a sum over the discrete values of the time-like component:

$$\int_p \to \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d\mathbf{p}}{(2\pi)^3} \ . \tag{61}$$

Unfortunately the simplicity of perturbation theory in the imaginary time formalism is somewhat offset by the fact that, when one is interested in the answers to dynamical questions, it is necessary to continue results back to real time. However, the effective potential, which is the generating functional for Green's functions at zero momentum, can be computed in the imaginary time formalism because at zero momentum (static physics) there is no difference between real time and imaginary time. For example for the $\lambda\Phi^4$ scalar theory both in imaginary time and in real time one finds that the one-loop effective potential is given by

$$V(\phi) = V_{cl}(\phi) + \frac{1}{\beta}\sum_n \int \frac{d\mathbf{k}}{(2\pi)^3} \ln(\frac{4\pi^2 n^2}{\beta^2} + \mathbf{k}^2 - a + \frac{\lambda}{2}\phi^2) \ . \tag{62}$$

4. Non-Equilibrium Quantum Field Theory

4.1. Isoentropic Energy-non-conserving Time Evolution

A physical system is described by its density matrix ρ,

$$\rho \equiv \frac{e^{-\beta H}}{tr\ e^{-\beta H}} \ , \tag{63}$$

and average values of observables O are determined by the density matrix:

$$< O >= tr\rho O \ . \tag{64}$$

Finite temperature field theory describes equilibrium physics, and therefore involves a time-independent density matrix. In general, however, the density matrix is time-dependent, and the task of non-equilibrium quantum field theory is to study the time evolution of ρ.

Non-equilibrium physics is a vast subject, and there is no canonical approach to its investigation. Usually the approach is suggested by the specifics of the physical system that one wants to describe. We discuss an approach which is useful in early universe cosmology, and is set up in the framework of the field theoretic Schrödinger picture, which is particularly suitable to time-dependent problems that require an initial condition for specific solution.

The functional density matrix is given by a superposition of wave functionals

$$\rho(\phi_1, \phi_2) = \sum_n p_n \Psi_n(\phi_1)\Psi_n^*(\phi_2) \ , \tag{65}$$

were $\{\Psi_n\}$ is a complete set of wave functionals, and p_n is the probability ($\sum_n p_n = 1$) that the system is in the state Ψ_n. In general both Ψ_n and p_n are time-dependent.

In equilibrium the dynamics is time-translation invariant and energy is conserved. The complete set of wave functionals $\{\Psi_n\}$ can be chosen to be the set of the (time-dependent) energy eigenstates, and the p_n are time-independent and are given by the canonical Boltzmann distribution:

$$p_n = \frac{e^{-\beta E_n}}{\sum_n e^{-\beta E_n}} \ , \tag{66}$$

where E_n is the energy eigenvalue of the state Ψ_n. The time evolution of the density matrix is trivial: it remains constant in time because both the p_n's and $\Psi_n\Psi_n^*$ are constant (N.B. the time dependence of the Ψ_n's is just a phase).

For non-equilibrium physics the time evolution of the density matrix is instead nontrivial. In fact, the p_n's need not be Boltzmann factors and can change in time, and it might not be possible to choose the Ψ_n's as energy eigenstates. We assume that the time dependence of the Ψ_n's be determined by the time-dependent Schrödinger equation. As a consequence, the density matrix ρ satisfies the following differential equation

$$\frac{d\rho}{dt} = \sum_n p_n \frac{d}{dt}(\Psi_n\Psi_n^*) + \sum_n \frac{dp_n}{dt}(\Psi_n\Psi_n^*) = i[\rho, H] + \sum_n \frac{dp_n}{dt}(\Psi_n\Psi_n^*) \ . \tag{67}$$

In order for Eq.(67) to describe a well defined initial value problem for the time evolution of ρ, it is necessary to give the form of H and a model for dp_n/dt. We investigate Eq.(67) in the case of time independent p_n's (which corresponds to entropy-conserving time evolution) and time dependent Hamiltonian* (which corresponds to energy non-conserving time evolution)[16]. With these hypotheses Eq.(67) takes the form of the quantum Liouville-von Neumann equation with time dependent Hamiltonian

$$\frac{d\rho}{dt} = i[\rho, H] \ . \tag{68}$$

In particular, we assume that the "interesting" time dependence of H occurs in an interval $t_i < t < t_f$ whereas $H = H_i = constant$ when $t < t_i$ and $H = H_f = constant$ when $t > t_f$. Interesting Hamiltonians, which may be investigated, include Hamiltonians with time-dependent mass squared (in particular the mass squared changing sign is useful in the study of symmetry-changing phase transitions), and Hamiltonians corresponding to a quantum field theory in a background Robertson-Walker metric with time-dependent scale factor.

We impose as initial condition that the density matrix ρ when $t \leq t_i$ be given by the Boltzmann distribution for H_i with temperature T_i. The typical result, which one seeks using the formalism that we developed, is the density matrix at times $t > t_f$, *i.e.* one solves Eq.(68) and examines ρ at late times. If ρ still changes with time when $t > t_f$, one concludes that the system remains out of equilibrium. If ρ is time-independent for $t > t_f$, there are two possibilities: either ρ is given by a Boltzmann distribution for H_f with a temperature T_f, in which case the system regains thermal equilibrium at temperature T_f, or ρ is not given by a Boltzmann distribution, in which case the system reaches non-thermal equilibrium.

4.2. Methods for Solving the Liouville-von Neumann equation

We are interested in obtaining solutions to the Liouville-von Neumann equation (68) with a canonical density matrix as initial condition. However, the exact solution

* The physical idea behind these assumptions is that one drops the time dependence of the p_n's, in order to make the equations treatable, but tries to account for (part of) this time dependence by introducing an "effective" time-dependent Hamiltonian.

of this problem is only feasible[16] for problems that are described by a quadratic Hamiltonian. For more general Hamiltonians one can obtain approximate solutions using the observation that the Liouville-von Neumann equation can be derived from a variational principle introduced by Balian and Veneroni[17]. An approximate application of this variational principle with a restricted variational *Ansatz* leads to approximate equations for the density matrix that may be integrated.

The Liouville-von Neumann equation can be derived by varying the actionlike quantity

$$I = -\int_{t_i}^{t_f} dt\ tr\left[\rho\left(\frac{d\Lambda}{dt} + i[H,\Lambda]\right)\right] - [tr(\rho\Lambda)]_{t=t_i}\ , \tag{69}$$

where Λ, ρ, and H are time-dependent kernels and the trace is over these kernels. Λ is a Lagrangian multiplier kernel. By varying I with respect to Λ and ρ one obtains respectively

$$\delta_\Lambda I = \int_{t_i}^{t_f} dt\ tr\left[\frac{d\rho}{dt} + i[H,\rho]\right]\delta\Lambda - [tr(\rho\delta\Lambda)]_{t=t_f}\ , \tag{70}$$

$$\delta_\rho I = -\int_{t_i}^{t_f} dt\ tr\left[\frac{d\Lambda}{dt} + i[H,\Lambda]\right]\delta\rho - [tr(\Lambda\delta\rho)]_{t=t_i}\ . \tag{71}$$

As in all time dependent variational principles we impose boundary conditions; we require that

$$[\delta\Lambda]_{t=t_f} = 0\ , \tag{72}$$

so we may set

$$[\Lambda]_{t=t_f} = identity\ . \tag{73}$$

Also we require, according to our program,

$$[\rho]_{t=t_i} = Boltzmann\ distribution\ , \tag{74}$$

and therefore

$$[\delta\rho]_{t=t_i} = 0\ . \tag{75}$$

Demanding that I be stationary against both variations of Λ and ρ, with the above boundary conditions, gives the Liouville-von Neumann equation for ρ, and also for Λ. The boundary condition Eq.(73) selects the static solution $\Lambda = identity$, for all time, so that Λ disappears from the discussion and we are left with a variational formulation of the Liouville-von Neumann equation for ρ.

If, rather than performing arbitrary Λ and ρ variations, we evaluate I with specific parameter dependent expressions for Λ and ρ and then vary these parameters, we obtain an approximate solution of the problem. In particular, a Gaussian *Ansatz* can be very useful. In this case ρ can be parametrized, also exploiting the Hermiticity of ρ $[\rho(\phi_1,\phi_2) = \rho^*(\phi_2,\phi_1)]$, as

$$\rho(\phi_1,\phi_2) = e^{-\gamma} exp\left\{-\frac{1}{2}\left[\phi_1\left(\frac{G^{-1}}{2} - 2i\Pi\right)\phi_1 + \phi_2\left(\frac{G^{-1}}{2} + 2i\Pi\right)\phi_2 - \phi_1\left(G^{-1/2}\xi G^{-1/2}\right)\phi_2\right]\right\}\ , \tag{76}$$

where G and Π are real and symmetric kernels, ξ is Hermitian, and γ is to be chosen so that ρ is properly normalized. ξ is called the *degree of mixing*; in fact, it is a measure of the amount by which ρ differs from a pure state. For $\xi = 0$, $\rho = \Psi(\phi_1)\Psi^*(\phi_2)$, with

$$\Psi(\phi) = (det^{-1/4} 2\pi G) exp\left[-\frac{1}{2}\phi\left(\frac{G^{-1}}{2} - 2i\Pi\right)\phi \right] . \tag{77}$$

The meaning of G and Π can be understood by looking at the following bilinear expectation values

$$\begin{aligned} &< \Phi(\mathbf{r})\Phi(\mathbf{r}') > = [G^{1/2}(1-\xi)^{-1}G^{1/2}](\mathbf{r},\mathbf{r}') \\ &< \Pi(\mathbf{r})\Pi(\mathbf{r}') > = \frac{1}{4}[G^{-1/2}(1+\xi)G^{-1/2}](\mathbf{r},\mathbf{r}') + 4[\Pi G^{1/2}(1-\xi)^{-1}G^{1/2}\Pi](\mathbf{r},\mathbf{r}') \\ &< \Phi(\mathbf{r})\Pi(\mathbf{r}') > = \frac{i}{2}\delta(\mathbf{r}-\mathbf{r}') + 2[G^{1/2}(1-\xi)^{-1}G^{1/2}\Pi](\mathbf{r},\mathbf{r}') \ . \end{aligned} \tag{78}$$

This approximation scheme has been used in an investigation of early universe evolution[16]. The results support the validity of the heuristic method of analysis of early universe phase transitions (a dynamical problem) in which one evaluates only the (static) effective potential and assumes that the evolution be adequately described by substituting the effective potential in place of the classical potential in classical evolution equations[18].

5. Chern-Simons Theory

5.1. Physical Relevance of 3-dimensional Quantum Field Theories

In this section we discuss a particular 3-dimensional quantum field theory. There are several physical circumstances in which a 3-dimensional quantum field theory can be relevant.

A first example is given by Minkowski 4-dimensional field theory in the Schrödinger Hamiltonian formalism, wherein one works at fixed time and therefore the c-number arguments of the wave functionals depend only on the three spatial coordinates, they are functionals of 3-dimensional fields.

Another example is given by situations in which the interesting physics is (at least approximately) confined to a plane. This scenario occurs in some condensed matter phenomena, like the quantum Hall effect and high T_c superconductivity, and for motion in presence of cosmic strings, which is described by planar gravity.

Yet another circumstance in which 3-dimensional field theories are relevant is directly related to the topic of this workshop. In fact, one expects that in the high temperature limit thermal field theory in four dimensions can be described by a 3-dimensional field theory. A naive formal argument, which is often used to support this expectation, is based on the observation that, because of the periodic boundary conditions imposed on the fields, finite temperature field theory in the imaginary time formalism "lives" in $R^3 \times S^1$, and the radius of the S^1 is given by $1/T$. In the limit $T \to \infty$ the S^1 collapses leaving a 3-dimensional space R^3. This apparent dimensional reduction in the high temperature limit can be also seen in

Feynman diagrams; for example in the $\lambda\Phi^4$ scalar theory the tadpole contribution to the self-energy is

$$iT\sum_n \int \frac{d\mathbf{p}}{(2\pi)^3}\ \lambda\ \frac{i}{-4n^2\pi^2T^2-\mathbf{p}^2-m^2}\ , \tag{79}$$

and its high temperature limit, in which only the n=0 term of the sum survives, is given by

$$\int \frac{d\mathbf{p}}{(2\pi)^3}\ \lambda T\ \frac{1}{\mathbf{p}^2+m^2}\ . \tag{80}$$

The passage from Eq.(79) to Eq.(80) can be interpreted again as a dimensional reduction (and a dimensionful scaling of the coupling $\lambda \to \lambda T$).

This possible thermal field theory application was one of the initial motivations for studying the specific 3-dimensional field theory, involving the Chern-Simons term, which we review in the following. Recent results[19] in the context of the *hard loops* high temperature approximation have found that a quantity which naturally appears in the context of Chern-Simons theories is indeed directly related to the generating functional for hard thermal loops in QCD.

5.2. The Chern-Simons Term

The Chern-Simons term* $\Omega(A)$ is a structure which can be constructed out of gauge fields in three dimensions:

$$\Omega(A) \equiv -\frac{1}{8\pi^2}\epsilon^{ijk}tr(\partial_i A_j A_k + \frac{2}{3}A_i A_j A_k)\ . \tag{81}$$

(If the gauge fields are Abelian the last term on the right is absent.) $\Omega(A)$ is a topological quantity, it is independent of the metric.

Even though it is a 3-dimensional object, the Chern-Simons term appears in the study of some 4-dimensional problems. It arose first in the Schrödinger-picture description of the vacuum angle of quantized 4-dimensional non-Abelian gauge theories. This comes about in the following manner. Consider the quantity

$$W(A) \equiv \int d\mathbf{x}\ \Omega(A)\ , \tag{82}$$

which satisfies Gauss law, *i.e.*

$$\left(D_i\frac{\delta}{\delta A_i}\right)_a W(A) = D_i\left(\frac{1}{16\pi^2}\epsilon^{ijk}F^a_{jk}\right) = 0\ . \tag{83}$$

However $W(A)$ is not gauge invariant: under a finite gauge transformation, corresponding to the gauge group element g, $W(A)$ changes by the winding number n_g of the gauge transformation:

$$W(A^g) - W(A) = \int d\mathbf{x}\ [\Omega(A^g) - \Omega(A)] = \int d\mathbf{x}\ \frac{1}{8\pi^2}\epsilon^{ijk}[\partial_i(\partial_j g\ g^{-1}A_k) + \omega(g)] = n_g\ , \tag{84}$$

* For a review, see Ref.[20].

where

$$\omega(g) \equiv \frac{1}{24\pi^2}\epsilon^{ijk} tr(g^{-1}\partial_i g \; g^{-1}\partial_j g \; g^{-1}\partial_k g) \; , \tag{85}$$

$n_g \equiv \int d\mathbf{x} \; \omega(g)$, and we assume that the gauge field vanishes sufficiently rapidly as $|\mathbf{x}| \to \infty$ so that the surface term, which arises from $\int d\mathbf{x} \; \epsilon^{ijk}\partial_i(\partial_j g g^{-1} A_k)$, also vanishes. It is also true that $\omega(g)$ can be written as a total derivative, *i.e.* there exist some $\omega^i(g(q))$, where $q(x)$ is a parametrization for $g(x)$, such that

$$\omega(g) = \partial_i \omega^i(g(q)) \; . \tag{86}$$

However, because of the specific behavior of the parametrization q of the group elements as $|\mathbf{x}| \to \infty$, the integral of $\omega(g)$ over the whole volume is not zero, in spite of the fact that, owing to Eq.(86), it can be presented as an integral over the surface at infinity. Rather the integral takes an integer value n_g, which characterizes the homotopic equivalence class to which g belongs.

The relevance of all this to 4-dimensional QCD can be seen in the Schrödinger representation, where states are 3-dimensional functionals that must satisfy Gauss law. Therefore they can take the form

$$\Psi(A) = e^{i\theta W(A)}\psi(A) \; , \tag{87}$$

where $\psi(A)$ is gauge invariant, and consequently obeys Gauss law. The states $\Psi(A)$ also satisfy Gauss law but they are not gauge invariant. Rather under a gauge transformation g, belonging to the homotopy class n_g, they change according to

$$\Psi(A) \to \Psi(A^g) = e^{-i\theta n_g}\Psi(A) \; , \tag{88}$$

This is the origin of the famous vacuum θ-angle.

In 3-dimensional physics, the Chern-Simons term can be added to the ordinary Lagrange density of Yang-Mills theories leading to the Lagrange density of the *topologically massive theories*, which is given by

$$\mathcal{L}_m = \frac{1}{2} tr F^{\mu\nu} F_{\mu\nu} + 8\pi^2 k \Omega(A) \; . \tag{89}$$

$\mathcal{L}_m$ is not gauge invariant; however, at the classical level $\mathcal{L}_m$ describes a gauge invariant theory because the equations of motion that follow from $\mathcal{L}_m$ are gauge invariant:

$$D_\mu F^{\mu\nu} + \frac{k}{2}\epsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0 \; . \tag{90}$$

At the quantum level, the quantity that defines the field theory is the phase exponential of the action, and one sees[20] that it is gauge invariant for all values of k when the gauge group is Abelian, whereas in the non-Abelian case gauge invariance requires that k be quantized as follows: $4\pi k = integer$.

Interestingly, one can show that the gauge bosons described by $\mathcal{L}_m$ have mass k (therefore these theories, like the Higgs models, are counterexamples to the naive statement that gauge invariance requires gauge bosons to be massless), and this is the reason for the name topologically massive theories.

$\Omega(A)$ is also related to the Pontryagin density P. In fact,

$$P = -\frac{1}{32\pi^2} tr\epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = \partial_\alpha \Omega^\alpha(A) \ , \tag{91}$$

where

$$\Omega^\alpha(A) = -\frac{1}{8\pi^2} \epsilon^{\alpha\mu\nu\beta} tr(\partial_\mu A_\nu A_\beta + \frac{2}{3} A_\mu A_\nu A_\beta) \ , \tag{92}$$

and clearly, for fixed α, $\Omega^\alpha(A)$ can be identified with $(\pm)\Omega(A)$ if the dependence of the gauge fields on the α-th coordinate is suppressed.

Numerous mathematically interesting applications have been found for the Chern-Simons term; for example, the correlators of Wilson lines in a pure Chern-Simons theory (the theory discussed in the following subsection) are related to the polynomial invariants of knot theory[21].

5.3. Pure Chern-Simons Theory

The mathematical structure which is important in the context of the hard loops high temperature approximation arises in *pure Chern-Simons theory*. The Lagrange density $\mathcal{L}_{CS}$ of this theory is obtained as the $k \to \infty$ limit of $\mathcal{L}_m$, *i.e.* by neglecting in $\mathcal{L}_m$ the Yang-Mills kinetic term. Therefore $\mathcal{L}_{CS}$ is given by

$$\mathcal{L}_{CS} = 8\pi^2 k \Omega(A) \ . \tag{93}$$

The equation of motion which follows from $\mathcal{L}_{CS}$ is

$$\epsilon^{\alpha\mu\nu} F_{\mu\nu} = 0 \ . \tag{94}$$

Classically this theory is trivial: the solutions of the equation of motion satisfy $F_{\mu\nu} = 0$ and they are therefore pure gauge. However, as a quantum field theory this simple model can have some interesting structure. We analyze the theory as a canonical quantum field theory in the gauge $A_0 = 0$; the Lagrange density is

$$\mathcal{L}_{CS} = \frac{k}{2} \epsilon^{ij} tr \dot{A}_i A_j \ . \tag{95}$$

By varying with respect to A_i one obtains the equations of motion

$$\dot{A}_i = 0 \ . \tag{96}$$

The Hamiltonian vanishes: $H = 0$. Because of the choice $A_0 = 0$, the equation gotten from Eq.(93) by varying with respect to A_0 is not obtained; rather a constraint is imposed

$$G^a \equiv -\frac{k}{2} \epsilon^{ij} F^a_{ij} = 0 \ . \tag{97}$$

This is the present analog of the Gauss law constraint in ordinary Yang-Mills theory. (One easily verifies that the G^a's are the generators of static gauge transformations.)

There are two ways of quantizing this theory: one can either solve the constraint (97) first and then quantize the remaining degrees of freedom or quantize first and

then solve the constraint. If one solves the constraint first the vector potentials are pure gauges, and if the topology of the space is trivial no interesting structure arises in the theory. We follow the alternative strategy, *i.e.* quantize before solving the constraint, which does lead to some interesting structures.

The Lagrange density $\mathcal{L}_{CS}$ involves only first order time derivatives; for this reason, unlike what happens in Yang-Mills theories, the components of the vector potential do not commute:

$$[A_i^a(\mathbf{x}), A_j^b(\mathbf{y})] = \frac{i}{k}\epsilon_{ij}\delta(\mathbf{x}-\mathbf{y}) \ . \tag{98}$$

Since $H = 0$, there is no dynamics, and all the interesting aspects of the theory come from the solution of the constraint, *i.e.* by requiring that the generators $G^a(\mathbf{x})$ annihilate the physical states

$$G^a(\mathbf{x})|\Psi>=0 \ . \tag{99}$$

We solve Eq.(99) in the Schrödinger picture. Because the spatial components of the vector potential do not commute, the wave functionals depend on just one of the two spatial components. [Instead for 3-dimensional Yang-Mills theories in the (2-dimensional) fixed-time Schrödinger picture, the wave functionals depend on both spatial components.] We choose A_1^a, which we call ϕ^a, as the argument of the the wave functionals $\Psi(\phi)$, and A_2^a is realized by functional differentiations with respect to ϕ^a,

$$\begin{aligned} |\Psi> &\sim \Psi(\phi) \\ A_1^a(\mathbf{x})|\Psi> &\sim \phi^a(\mathbf{x})\Psi(\phi) \\ A_2^a(\mathbf{x})|\Psi> &\sim \frac{1}{ik}\frac{\delta}{\delta\phi^a(\mathbf{x})}\Psi(\phi) \ , \end{aligned} \tag{100}$$

which is consistent with the commutation relations in Eq.(98).

In our realization of the Schrödinger picture, the constraint equation (99) can be written in the form of functional differential equation

$$\left(\partial_1\frac{\delta}{\delta\phi^a(\mathbf{x})} + f^{abc}\phi^b(\mathbf{x})\frac{\delta}{\delta\phi_c(\mathbf{x})} - ik\partial_2\phi^a(\mathbf{x})\right)\Psi(\phi) = 0 \ . \tag{101}$$

We solve the Gauss law constraint in two steps. First we determine the result of a finite gauge transformation g, implemented by the unitary operator $U(g)$, on a state, *i.e.* we evaluate

$$U(g)\Psi(\phi) = e^{i\int_x \lambda^a(x)G^a(x)}\Psi(\phi) \equiv e^{iG}\Psi(\phi) \ , \tag{102}$$

[in Eq.(102) we also indicate the explicit form of $U(g)$ in terms of the generators G^a, and thereby define the operator G], and then we demand that

$$U(g)\Psi(\phi) = e^{iG}\Psi(\phi) = \Psi(\phi) \ , \tag{103}$$

as required by the Gauss law constraint $G^a|\Psi>=0$.

Let us observe that

$$G \equiv \int_x \lambda^a G^a = -\int_x \lambda^a \left(\partial_1 \frac{1}{i} \frac{\delta}{\delta \phi^a} + f^{abc} \phi^b \frac{\delta}{\delta \phi_c} \right) - k \int_x \phi^a \partial_2 \lambda^a \equiv G_\phi + 2k \int_{\mathbf{x}} tr \phi \partial_2 \lambda \ , \qquad (104)$$

where G_ϕ is the generator of infinitesimal gauge transformations on the argument $\phi^a = A_1^a$ of the wave functional. Due to the presence of the term $2k \int_{\mathbf{x}} tr\phi\partial_2\lambda$, the full action of a gauge transformation on the state is realized with a 1-cocycle (see Ref.[22]):

$$U(g)\Psi(\phi) \equiv e^{iG}\Psi(\phi) = e^{iG} e^{-iG_\phi} e^{iG_\phi} \Psi(\phi) = e^{iG} e^{-iG_\phi} \Psi(\phi^g) = e^{-2\pi i \alpha_1(\phi;g)} \Psi(\phi^g) \ , \qquad (105)$$

where $\phi^g \equiv g^{-1}\phi g + g^{-1}\partial_1 g$, and α_1 is obtained by evaluating $e^{iG}e^{-iG_\phi}$ with the help of the Baker-Hausdorff procedure

$$\alpha_1(\phi;g) \equiv -\frac{k}{2\pi} \int_x tr(2\phi\partial_2 g \ g^{-1} + g^{-1}\partial_1 g \ g^{-1}\partial_2 g) + 4\pi k \int_x \omega^0(g) \ . \qquad (106)$$

The ω^0 which appears in Eq.(106) is the time-like component of the 3-vector ω^μ, which we already encountered in Eq.(86), and one can obtain an explicit expression for ω^0 only after a parametrization for $g(x)$ has been chosen. Alternatively $\int_x \omega^0(g)$ may be presented as the integral of $\omega(g) \equiv \partial_\mu \omega^\mu(g)$ over a three-manifold M, whose boundary is the two-space on which our Schrödinger functional is defined.
$\alpha_1(\phi;g)$ is a one-cocycle because it satisfies the condition

$$\alpha_1(\phi;g) = \alpha_1(\phi;g\tilde{g}) - \alpha_1(\phi^g;\tilde{g}) \ , \qquad (107)$$

which is called the one-cocycle condition.

Eq.(105) concludes the first step in our strategy for solving the Gauss law constraint; at this point we demand that $U(g)\Psi(\phi) = \Psi(\phi)$, and find that the physical wave functionals are not gauge invariant, but rather they satisfy

$$\Psi(\phi^g) = e^{2\pi i \alpha_1(\phi;g)} \Psi(\phi) \ . \qquad (108)$$

A general wave functional which verifies Eq.(108) can be written in the form

$$\Psi(\phi) = e^{iW(\phi)} \psi(\phi) \ . \qquad (109)$$

where $\psi(\phi)$ is completely gauge invariant $[\psi(\phi^g) = \psi(\phi)]$, $W(\phi)$ is given by

$$W(\phi) \equiv -k \int_x tr(h^{-1}\partial_1 h \, h^{-1}\partial_2 h) + 8\pi^2 k \int_x \omega^0(h) \ , \qquad (110)$$

and h is to be expressed in terms of ϕ using $\phi \equiv h^{-1}\partial_1 h$. $W(\phi)$ is a *cochain*, it satisfies $W(\phi) = W(\phi^g) - 2\pi\alpha_1(\phi;g)$.

The term $4\pi k \int_x \omega^0(h)$ is multivalued, and as a consequence it turns out that the in order for the wave functional in Eq.(109) to be single valued one needs to require that $4\pi k$=integer. This is the way that the quantization condition, which we

obtained by requiring the gauge invariance of the phase exponential of the action, arises in the present, Schrödinger-picture formalism.

Note that on a topologically trivial space, one cannot construct a gauge invariant functional depending only on $\phi^a = A_1^a$. Consequently $\psi(\phi)$ in Eq.(109) is a constant N, and the general state of Chern-Simons theory is just

$$\Psi(\phi) = Ne^{iW(\phi)} \ . \tag{111}$$

In conclusion, it is interesting to verify that Eq.(97) has indeed been satisfied. To this end we consider $A_i\Psi(\phi)$. For $i = 1$, we have the parametrization (by definition) $A_1 = \phi = h^{-1}\partial_1 h$. Hence $A_1\Psi(\phi) = h^{-1}\partial_1 h\Psi(\phi)$. For $i = 2$, $A_2 = (1/i)\delta/\delta\phi^a$ and $A_2^a\Psi(\phi) = (\delta W(\phi)/\delta\phi^a)\Psi(\phi)$. While W is presented in Eq.(110) as a functional of h, we may use the chain rule, and the above formula relating ϕ to h. In this way one finds $A_2(\phi)\Psi(\phi) = h^{-1}\partial_2 h\Psi(\phi)$ and one concludes that, acting on states, A_i is a pure gauge

$$A_i\Psi(\phi) = h^{-1}\partial_i h\Psi(\phi) \ . \tag{112}$$

Therefore, the corresponding field strength vanishes, in keeping with Eq.(97).

5.4. The Chern-Simons Eikonal

In 1+1-dimensional point-particle quantum mechanics with Lagrangian

$$L = \frac{m\dot{x}^2}{2} - V(x) \ , \tag{113}$$

and energy constraint

$$\frac{p^2}{2m} + V(x) = E \ , \tag{114}$$

the *eikonal*, which is the exponent of the WKB wave functions, is given by

$$W(x) \equiv \int^x dx' \ p(x') \ , \tag{115}$$

where $p(x)$ is the solution of Eq.(114): $p(x) = \sqrt{2m[E - V(x)]}$.

In Chern-Simons theory the Lagrangian can be written as

$$L_{CS} = k \ tr\dot{A}_1 A_2 \ , \tag{116}$$

the Hamiltonian vanishes, and there is the constraint

$$F_{12}^a = \partial_1 A_2^a - \partial_2 A_1^a + f^{abc}A_1^b A_2^c = 0 \ . \tag{117}$$

Clearly there is an analogy with the 1+1-dimensional point-particle quantum mechanics based on the identifications $A_1 \sim x$ and $kA_2 \sim p$. One can therefore introduce WKB wave functional

$$\Psi(\phi) = \exp\left(i\int^{\phi^a} DA_1 \ k \ A_2^a(A_1)\right) \equiv \exp iW(\phi) \ , \tag{118}$$

where $A_2^a(A_1)$ is to be obtained by solving Eq.(117).

An explicit solution $A_2^a(A_1)$ of Eq.(117) is obtained perturbatively in Ref.[19] as a series in A_1. Here, we observe that $W(\phi)$ coincides with the phase exponential of the wave functional that solves the Gauss law constraint (Eqs.(109)-(110)), *i.e.* the WKB/eikonal wave functional is exact in Chern-Simons theory, owing to the simple dynamical structure of that theory. To see the result, observe that $W(\phi)$ defined by Eq.(118) satisfies

$$\frac{\delta W(\phi)}{\delta \phi^a} = k\, A_2^a \ . \tag{119}$$

and as a consequence of Eq.(117), $W(\phi)$ also satisfies

$$\partial_1 \frac{\delta W(\phi)}{\delta \phi^a(\mathbf{x})} + f^{abc}\phi^b(\mathbf{x})\frac{\delta W(\phi)}{\delta \phi_c(\mathbf{x})} - k\partial_2\phi^a(\mathbf{x}) = 0 \ . \tag{120}$$

This shows that the wave functional in Eq.(118) satisfies Eq.(101).

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